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FREE VIBRATIONS FOR A SEMILINEAR WAVE EQUATION

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ABSTRACT

Existence and regularity of nontrivial time periodic solutions are proved for semilinear wave equations of the form

$$u_{tt} - u_{xx} + f(x,u) = 0 \qquad 0 < x < \pi, \; t \in \mathbb{R}$$
$$u(x,b) = 0 = u(\pi,b)$$

under mild smoothness, monotonicity, and superlinearity assumptions on f. The forced vibration case where f depends on b in a time periodic fashion is also treated.

AMS(MOS) Classification No. 35L05, 35L60, 35B10, 49G99

Key Words: Nonlinear wave equation, periodic solution, free vibrations, variational methods, critical points.

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FREE VIBRATIONS FOR A SEMILINEAR WAVE EQUATION

Paul H. Rabinowitz

Introduction

This paper is primarily concerned with the existence and regularity of free vibrations for a one dimensional semilinear wave equation. To be more precise, consider the partial differential equation:

$$(0.1) \qquad u_{tt} - u_{xx} + f(x,u) = 0, \quad x \in (0,\pi), \ t \in \mathbb{R}$$

together with the boundary conditions

$$(0.2) \qquad u(0,t) = 0 = u(\pi,t), \qquad t \in \mathbb{R}.$$

We are interested in the existence of solutions of (0.1)-(0.2) which are periodic in t. Suppose $f(x,0) \equiv 0$ so $u \equiv 0$ is such a solution. Following the terminology used in ordinary differential equations, we call a nontrivial time periodic solution of (0.1)-(0.2) a free vibration. One of the difficulties in free vibration problems is that the period is not known a priori. Our main result is that under conditions on f given in §1, for any period which is a rational multiple of π, (0.1)-(0.2) possesses a classical free vibration possessing that period.

While our major concern is with free vibrations, the techniques we use for (0.1)-(0.2) work equally well for the forced vibration case where f depends explicitly on t in a periodic fashion as well as on x and u.

There has not been much work on periodic solutions of such wave equations. A certain amount of effort has gone into the study of perturbation problems for the forced vibration case where f is replaced by $\varepsilon g(x,t,u)$ in (0.1) with ε near 0. See e.g. [1] and the references cited there. Very little has been done for the free vibration case. A few people have given formal solutions for

$$(0.3) \qquad u_{tt} - u_{xx} + \varepsilon g(x,u,u_x,u_t) = 0$$

together with boundary and periodicity conditions. See [2]. Kurzweil [3]-[4] used an averaging method for a problem of the form (0.3) with u_t and u_x terms appearing in a special fashion.

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More recently Fink, Hall, and Hausrath [5] and Stedry and Vejvoda [2] have obtained some results for equations of the form (0.3) with such g's as $(1 - u_t^2)u_t$, $(1 - u^2)u_t$, and $(1 - u^2)u$. They obtain weak solutions which are piecewise continuous. Melrose and Pemberton [6] have obtained continuous weak solutions for an equation like (0.3). There does not seem to have been any rigorous work other than our own on (0.1)-(0.2) for nonperturbation problems.

Naturally one must use global methods to treat (0.1) for f's which are not small. Our approach to (0.1)-(0.2) is via the calculus of variations. For definiteness suppose we are trying for a 2π periodic solution of (0.1)-(0.2) and f depends only on u. Let $T = [0,\pi] \times [0,\pi]$ and consider

$$\Phi(u) = \int_T [\tfrac{1}{2}(u_t^2 - u_x^2) - F(u)]dxdt \tag{0.4}$$

where

$$F(z) = \int_0^z f(s)ds .$$

Thus the integrand for (0.4) is the Lagrangian for our problem. Formally critical points of Φ defined on a suitable class of t periodic functions are weak solutions of (0.1)-(0.2). The function Φ is indefinite in the sense that it is neither bounded from above or from below on e.g. $W^{1,2}(T)$. We do not know how to obtain nontrivial critical points of Φ in any direct fashion. However under conditions on f given in §1, nontrivial critical points u_n can be obtained for Φ restricted to finite dimensional subspaces E_n of admissible functions. Difficulties which arise in attempting to get the functions u_n to converge to a solution of (0.1)-(0.2) lead us to study a modified Lagrangian and a corresponding modified version of (0.1). This is carried out in §1. Estimates are obtained in §2 which enable us to solve the modified problem. Then the existence of weak solutions of (0.1)-(0.2) is shown in §3 and their regularity is studied in §4.

For all of the above we assume $f = f(u)$ and we seek a 2π periodic solution in t of (0.1)-(0.2). Various extensions of our main results are carried our in §5. In particular we show there are free vibrations of (0.1)-(0.2) for any period which is a rational multiple of π. The effect of weakening the hypotheses made on f in §1 is also studied. Finally the forced vibration case is treated. Some topological results required in §1 are proved in the Appendix.

The variational method used in §1 to obtain approximate solutions is fairly general and may be useful in other situations involving variational problems with indefinite integrands.

A natural further question to pursue is whether results analogous to ours can be obtained for free vibrations of quasilinear wave equations such as

$$u_{tt} - (\sigma(u_x))_x + f(u) = 0 \tag{0.5}$$

together with (0.2). Quite recently R. Di Perna has proved that in striking contrast to (0.1), if $f \equiv 0$ and σ satisfies some reasonable conditions, then (0.5), (0.2) possesses no nontrivial free vibrations.

A second interesting question to ask and for which we have no answers is whether (0.1)-(0.2) possesses free vibrations with periods which are irrational multiples of π.

Finally we are indebted to E. Fadell for his assistance with the topological result, Lemma A.2, given in the Appendix. We also thank L. Nirenberg for some ideas which led to the proof of the regularity result, Theorem 4.1, and to H. Brezis for some suggestions which led to Theorem 5.6.

§1. The modified problem and its approximate solution

Some notation is in order. Let $T = \{(x,t) | x \in [0,\pi], t \in [0,2\pi]\}$ and consider the set of C^∞ functions on T, 2π periodic in t and vanishing near $x = 0$ and $x = \pi$. Let $\dot{H}_m$ denote the closure of this set with respect to $\|u\|_{H_m} \equiv \sum_{|\sigma| \leq m} \|D^\sigma u\|_{L^2}$ where $L^2 \equiv L^2(T) \equiv \dot{H}_0$, D^σ denotes a derivative of order $|\sigma|$ and the usual multi-index notation is being employed. Let H_m denote the closure of C^∞ functions on T, 2π periodic in t, with respect to $\|\cdot\|_{H_m}$. Let $C^m \equiv C^m(T)$ denote the set of m times continuously differentiable functions on T, 2π periodic in t, and satisfying the boundary conditions (0.2). The usual maximum norm is used for C^m.

We begin our investigation of (0.1)-(0.2) with the problem

$$(1.1) \qquad \begin{cases} u_{tt} - u_{xx} + f(u) = 0, & 0 < x < \pi, \; t \in \mathbb{R} \\ u(0,t) = 0 = u(\pi,t), & u(x,t+2\pi) = u(x,t) \end{cases}$$

which is technically slightly simpler than (0.1). It is assumed that f satisfies

(f_1) $\quad f \in C^1(\mathbb{R},\mathbb{R})$ and $f(0) = 0$

(f_2) $\quad f$ is strictly monotonically increasing, i.e., $z_1 > z_2$ implies $f(z_1) > f(z_2)$

(f_3) $\quad f$ is superlinear at 0 and ∞, i.e.

(i) $f(z) = o(|z|)$ at $z = 0$ and

(ii) There are constants $\bar{z} > 0$ and $\theta \in [0,\frac{1}{2})$ such that

$$F(z) = \int_0^z f(s)ds \leq \theta\, z\, f(z) \quad \text{for } |z| \geq \bar{z}.$$

<u>Remark 1.2:</u> By (f_2), $z f(z) > 0$ for $z \neq 0$ and by (f_3) (ii), if $|z| > \bar{z}$,

$$(1.3) \qquad \frac{1}{\theta z} \leq \frac{F'(z)}{F(z)}.$$

Integration and exponentiation of (1.3) then shows $|z|^{\frac{1}{\theta}} \leq \alpha F(z)$ for $|z| > \bar{z}$ where $\alpha > 0$. Hence by (f_3)(ii),

$$(1.4) \qquad |z|^{\frac{1}{\theta}-1} \leq \alpha\theta\,|f(z)| \qquad \text{for } |z| > \bar{z}$$

which justifies the term superlinear in (f_3)(ii).

Let $\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$. An important role in the study of (1.1) is played by $N(\Box)$, the null space of $\Box$. It is easy to see that the closure in L^2 of $N(\Box)$ is

$$N \equiv \{p(x+t) - p(-x+t) \mid p \in L^2(S^1)\} \tag{1.5}$$

where S^{k-1} denotes the unit sphere in $\mathbb{R}^k$. Let $N^\perp$ denote the orthogonal complement of N in L^2. If u is a classical solution of (1.1), then $u = v + w$, $v \in N$, $w \in N^\perp$. We can now state our main result:

Theorem 1.6: Let f satisfy (f_1)-(f_3) and $f \in C^k$, $k \geq 2$. Then there is a $u = v+w \in (C^k \cap N) \oplus (C^{k+1} \cap N^\perp)$ such that u is a nontrivial solution of (1.1).

The proof of Theorem 1.6 will be carried out in §1-4. It is a consequence of an existence result proved in §1-3 and a regularity theorem given in §4.

The fact that there is no upper bound on the growth of f as $|z| \to \infty$ creates some difficulty for us later which we bypass by introducing a C^1 truncation f_K of f defined by

$$f_K(z) = \begin{cases} f(z) , & |z| \leq K \\ f(K) + f'(K)(z-K) + \rho(K)(z-K)^3 , & z > K \\ f(-K) + f'(-K)(z+K) + \rho(K)(z+K)^3, & z < -K \end{cases} \tag{1.7}$$

where $\rho(K)$ will be chosen appropriately later. That $f_K(z)$ grows like z^3 at ∞ is not crucial but there is a cut-off power beyond which our arguments fail.

We will solve (1.1) with the aid of a modified problem which can now be introduced. Let $\beta, K > 0$. The modified equation for $u = v+w \in N^\perp$ then is:

$$\begin{gathered} \Box u - \beta v_{tt} + f_K(u) = 0 , \quad 0 < x < \pi , \quad t \in \mathbb{R} \\ u(0,t) = 0 = u(\pi,t) ; \quad u(x,t + 2\pi) = u(x,t) . \end{gathered} \tag{1.8}$$

Our goal is to solve (1.8) for any β, $K > 0$ and then obtain a solution of (1.1) by choosing K appropriately large and letting $\beta \to 0$.

Let

$$I(u) = \int_T [\tfrac{1}{2}(u_t^2 - u_x^2 - \beta v_t^2) - F_K(u)]dxdt \tag{1.9}$$

where $F_K(u) = \int_0^z f_K(s)ds$. Then $I(u)$ is defined for all $u \in \mathring{H}_1$ and it is easily verified that formally, any critical point of I is a weak solution of (1.8). We do not know how to obtain critical points of I directly. However by restricting I to finite dimensional subspaces E_n of L^2 or $\overset{\circ}{H}_1$, corresponding critical points, u_n, of $I|_{E_n}$ are obtained which can be shown to converge, along a subsequence, to a nontrivial solution of (1.8). The machinery which is used to find the approximate critical points is provided by the next theorem. Below, $B_\rho = \{x \in \mathbb{R}^m \mid |x| < \rho\}$, $\mathbb{R}^j = \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_i = 0,\ j+1 < i \le m\}$ and $(\mathbb{R}^j)^\perp = \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_i = 0,\ 1 \le i \le j\}$.

Theorem 1.10: Let $J \in C^1(\mathbb{R}^m, \mathbb{R})$ and $I: \mathbb{R}^m \to \mathbb{R}$ where $J(x) \le I(x)$ for all $x \in \mathbb{R}^m$ and $I(x) \le 0$ for all $x \in \mathbb{R}^k$. If there are constants $R > r > 0$ such that $J > 0$ in $(B_r \setminus \{0\}) \cap (\mathbb{R}^k)^\perp$ while $I \le 0$ on $\mathbb{R}^m \setminus B_R$, then J has a critical point in $\{x \in \mathbb{R}^m \mid J(x) > 0\}$ and a corresponding critical value characterized by

$$c = \inf_{h \in \Gamma} \max_{x \in \mathbb{R}^{k+1} \cap \bar{B}_R} J(h(x)) > 0 \tag{1.11}$$

where $\Gamma = \{h \in C(\mathbb{R}^{k+1} \cap \bar{B}_R, \mathbb{R}^m),\ h(x) = x \text{ if } I(x) \le 0\}$.

Remark 1.12: The proof of Theorem 1.10 is contained in the Appendix. Note that the hypotheses of the theorem imply $x = 0$ is a critical point for J and that J possesses a positive maximum with a corresponding critical point in B_R. However in general the maximum of J is not the critical value given by (1.11). In particular for our application where $J = I$ as defined in (1.9), I is not bounded from above or from below in $\overset{\circ}{H}_1$, so there is probably no hope of getting a solution of (1.8) via maximizing I on finite dimensional subspaces of $\overset{\circ}{H}_1$.

To obtain approximate solutions of (1.8), define

$$E_n = \operatorname{span}\{\sin jx \sin kt,\ \sin jx \cos kt \mid 0 \le j, k \le n\}.$$

Since $\dim E_n = 2n^2 + n \equiv m$, we can identify E_n with $\mathbb{R}^m$. Set

$$N^+ = \operatorname{span}\{\sin jx \sin kt,\ \sin jx \cos kt \mid k > j\}$$

and

$$N^- = \operatorname{span}\{\sin jx \sin kt,\ \sin jx \cos kt \mid j > k\}.$$

Set $J = I$ as defined in (1.9) and consider $I|_{E_n}$. By (f_1) and (1.7), $I \in C^1(E_n, \mathbb{R})$ and by (f_2) and (1.7), $I \le 0$ on $\mathbb{R}^k \equiv (N \oplus N^-) \cap E_n$. Moreover by (1.7), if $u \in L^2$ and $\|u\|_{L^2} = 1$, $I(\alpha u) \to -\infty$ as $|\alpha| \to \infty$. Hence there is an $R = R(n)$ such that $I(u) \le 0$ if $u \in E_n$ and $\|u\|_{L^2} \ge R$. Lastly observe that for $r = r(n)$ sufficiently small, by $(f_3)(i)$, $I > 0$ in $(B_r \backslash \{0\}) \cap (\mathbb{R}^k)^\perp$ (where $(\mathbb{R}^k)^\perp = N^+ \cap E_n$).

Let $\mathbb{R}^{k+1} = \mathbb{R}^k \oplus$ span $\{\sin x \sin 2t\}$. Invoking Theorem 1.10, we have:

Lemma 1.13: $I|_{E_n}$ possesses a critical value $c_n = c_n(\beta, K)$ characterized as

$$(1.14) \qquad c_n = \inf_{h \in \Gamma_n} \max_{u \in \mathbb{R}^{k+1} \cap \bar{B}_R} I(h(u)) > 0$$

where $\Gamma_n = \{h \in C(\mathbb{R}^{k+1} \cap \bar{B}_R, E_n) \,|\, h(u) = u \text{ if } I(u) \leq 0\}$.

Remark 1.15: Actually to apply Theorem 1.10, all we need is $z f_K(z) \geq 0$ instead of (f_2). Moreover if f_K is even, we can drop (f_2) and weaken $(f_3)(i)$. See e.g. [7] or [8]. We also observe for later reference that $(f_3)(i)$ will not be used for any of the estimates obtained from this point on until §5 except for Lemmas 2.47 and 3.40.

Our goal now is to show that if u_n is a critical point of $I|_{E_n}$ with $I(u_n) = c_n$, then a subsequence of u_n converges to a solution $u = u(\beta, K)$ of (1.8). To achieve this, estimates are required for the functions u_n. The first two steps in obtaining suitable bounds are contained in the following result.

For the remainder of this paper, subscripted α's, γ's, M's, and A's repeatedly denote positive constants.

Lemma 1.16: There exist positive constants A_1, M_1 independent of n, β, and K and such that

$1^\circ \qquad c_n \leq A_1$

$2^\circ \qquad \|f_K(u_n)\, u_n\|_{L^1} \leq M_1$.

Proof: We exploit (1.14) to get 1°. Set $V_n = (N \oplus N^- \oplus \text{span}\{\sin x \sin 2t\}) \cap E_n$. Since $h(u) = u \in \Gamma_n$, by (1.14),

$$(1.17) \qquad c_n \leq \max_{u \in V_n \cap B_{R(n)}} I(u) \leq \max_{u \in V_n} I(u).$$

The right hand side of (1.17) is finite by (1.7) and the form of I. Suppose the maximum is achieved at $u = \bar{u}_n$. Each $u \in V_n$ can be written as $u = r(a \sin\xi \sin x \sin 2t + \cos\xi\, \varphi(x,t))$ where $\|\sin x \sin 2t\|_{L^2} = a^{-1}$, $\varphi \in (N \oplus N^-) \cap E_n$, $\|\varphi\|_{L^2} = 1$, $\xi \in [0, 2\pi)$, and $r = \|u\|_{L^2}$. Choosing $u = \bar{u}_n$ with L^2 norm $r = r(n)$, and observing from (1.14) that $I(\bar{u}_n) > 0$, we find

$$(1.18)\qquad \frac{r^2}{2}\cos^2\xi\int_T(\varphi_x^2-\varphi_t^2)dxdt+\frac{\beta}{2}r^2\cos^2\xi\int_T \bar{v}_{nt}^2\,dxdt$$
$$+\int_T F_K(\bar{u}_n)dxdt \le \frac{3r^2}{2}\sin^2\xi$$

where $\bar{u}_n = \bar{v}_n + \bar{w}_n \in N \oplus N^\perp$. Since all integrals on the left hand side of (1.18) are nonnegative, we conclude that

$$(1.19)\qquad \int_T F_K(\bar{u}_n)dxdt \le \frac{3}{2}r^2 .$$

Remark 1.2 implies that for $|z| \le K$

$$(1.20)\qquad F_K(z) = F(z) \ge \alpha_1|z|^{\frac{1}{\theta}} - \alpha_2$$

where α_1, α_2 are independent of K. By (1.7) for $z > K$

$$(1.21)\qquad F_K(z) = F(K) + f(K)(z-K) + \frac{f'(K)}{2}(z-K)^2 + \frac{\rho(K)}{4}(z-K)^4 .$$

In particular for $z \in [K,2K]$,

$$F_K(z) \ge F(K) \ge \alpha_1 K^{\frac{1}{\theta}} - \alpha_2$$

while for $2K \le z \equiv \gamma K$, $\gamma \ge 2$, we have

$$F_K(z) \ge \frac{\rho(K)}{4}(z-K)^4 \ge \frac{1}{4}(\gamma-1)^4K^4 = \frac{1}{4}\left(\frac{\gamma-1}{\gamma}\right)^4 z^4 \ge \frac{z^4}{2^6}$$

provided that $\rho(K) \ge 1$ which choice we henceforth make. Similar estimates hold for $z < 0$. Hence for all $z \in \mathbb{R}$,

$$(1.22)\qquad F_K(z) \ge \alpha_3|z|^\delta - \alpha_4$$

where $\delta = \min(4,\theta^{-1}) > 2$ and the constants α_3,α_4 are independent of K.

Using (1.19), (1.22), and the Hölder inequality, we see that

$$(1.23)\qquad \frac{3}{2}r^2 \ge \alpha_3\int_T|\bar{u}_n|^\delta dxdt - \alpha_5 \ge \alpha_6\left(\int_T \bar{u}_n^2\,dxdt\right)^{\frac{\delta}{2}} - \alpha_5 = \alpha_6 r^\delta - \alpha_6 .$$

Since $\delta > 2$, (1.23) gives an upper bound for r:

$$(1.24)\qquad r \le \hat{r}$$

where $\hat{r}$ is independent of n, β, or K. Returning to (1.17) and using the form of I, we get

$$(1.25)\qquad c_n \le I(\bar{u}_n) \le \frac{3}{2}r^2\sin^2\xi \le \frac{3}{2}\hat{r}^2 \equiv A_1$$

which proves 1°.

To verify 2°, observe first that

(1.26) $$c_n = I(u_n) = \int_T [\frac{1}{2}(u_{nt}^2 - u_{nx}^2 - \beta v_{nt}^2) - F_K(u_n)]dxdt .$$

Since u_n is a critical point for $I|_{E_n}$,

(1.27) $$I'(u_n)\varphi = 0 = \int_T (u_{nt}\varphi_t - u_{nx}\varphi_x - \beta v_{nt}\psi_t - f_K(u_n)\varphi)dxdt$$

for all $\varphi \in E_n$ where $\varphi = \psi + \chi$, $\psi \in N$, $\chi \in N^{\perp}$. Choosing $\varphi = u_n$ in (1.27) and forming (1.26) - $\frac{1}{2}$(1.27) yields

(1.28) $$c_n = \int_T (\frac{1}{2} f_K(u_n)u_n - F_K(u_n))dxdt .$$

A simple computation shows that (with $u = u_n$)

(1.29) $$\int_T (\frac{1}{2} f_K(u)u - F_K(u))dxdt = \int_{T_1(u)} (\frac{1}{2} f(u)u - F(u))dxdt$$

$$+ \int_{T_2(u)} [\frac{1}{2} f(K) - F(K) - \frac{1}{2}(u-K)f(K) + \frac{1}{2} K(u-K)f'(K) + \frac{\rho(K)}{2}K(u-K)^3 + \frac{1}{4}\rho(K)(u-K)^4]dxdt$$

$$+ \int_{T_3(u)} [\frac{1}{2}f(-K)(-K) - F(-K) - \frac{1}{2}(u+K)f(-K) - \frac{1}{2}K(u+K)f'(-K) - \frac{\rho(K)}{2}(u+K)^3 + \frac{1}{4}\rho(K)(u+K)^4]dxdt$$

where $T_1(u) = \{(x,t) \in T \,|\, |u(x,t)| \le K\}$, $T_2(u) = \{(x,t) \in T \,|\, u(x,t) > K\}$, and $T_3(u) = T\backslash(T_1(u) \cup T_2(u))$.

Using (f_3)(ii), it is easy to verify that there is a constant $\gamma_1 > 0$ and independent of K such that

(1.30) $$\int_T (\frac{1}{2}f_K(u)u - F_K(u))dxdt \ge -\gamma_1 + (\frac{1}{2} - \theta)\int_{T_1(u)} f(u)u\, dxdt$$

$$+ \int_{T_2(u)} [(\frac{1}{2} - \theta) f(K)K + \frac{\rho(K)}{2} K(u-K)^3 + \frac{1}{8}\rho(K)(u-K)^4]dx\, dt$$

$$+ \int_{T_3(u)} [(\frac{1}{2} - \theta) f(-K)(-K) + \frac{\rho(K)}{2}(-K)(u+K)^3 + \frac{1}{8}\rho(K)(u+K)^4]dxdt$$

provided that $\rho(K) \ge 4(f(K)^4 + f(-K)^4)$ which choice we make. Lastly there is a constant $\gamma_2 > 0$ and independent of K such that

(1.31) $$\int_T f_K(u)udxdt \le \gamma_2 + \int_{T_1(u)} f(u)u\, dxdt$$

$$+ \int_{T_2(u)} [f(K)K + \rho(K)K(u-K)^3 + \frac{1}{2}\rho(K)(u-K)^4]\, dxdt$$

$$+ \int_{T_3(u)} [-Kf(-K) + \rho(K)(-K)(u+K)^3 + \frac{1}{2}\rho(K)(u+k)^4]dxdt$$

provided that

$$\rho(K) \geq f(K)^4 + f'(K)^2 + K^4 f'(K)^4 + f(-K)^4 + f'(-K)^2 + k^4 f'(-K)^4$$

which we further assume. Combining (1.28), (1.30)-(1.31) yields

$$c_n + \gamma_3 \geq \gamma_4 \int_T f_K(u_n)u_n \, dxdt \tag{1.32}$$

for constants γ_3, γ_4 independent of n, β, and K. Hence 2° obtains.

Remark 1.33: For later reference, observe that

$$\|f_K(u_n)\|_{L^1} \leq M_2 \tag{1.34}$$

with M_2 independent of n, β, and K since (with $u = u_n$),

$$\int_T |f_K(u)| \, dxdt \leq \int_{T_4(u)} |f_K(u)| \, dxdt + \int_{T_5(u)} f_K(u)u \, dxdt \tag{1.35}$$

where $T_4(u) = \{(x,t) \in T \mid |u(x,t)| \leq 1\}$ and $T_5(u) = T \backslash T_4(u)$. Hence (1.34) follows from (1.35) and 2° of Lemma 1.16. Note also from the same result and the form of $f_K(u)$ that

$$\|u_n\|_{L^4} \leq A_2 \tag{1.36}$$

where A_2 is independent of n and β but depends on K.

Remark 1.37: It is not difficult to verify that to obtain the results of this section it suffices to assume (f_3) and

(f_1') $\quad f \in C(\mathbb{R},\mathbb{R})$ and $f(0) = 0$

(f_2') $\quad f$ is monotone nondecreasing

provided that we drop the $f'(K)$ term in (1.7).

§2. Solution of the modified problem.

Starting from the bounds just obtained for $f_K(u_n)$, we shall use bootstrap arguments to estimate u_n in stronger norms. These new estimates depend on β and K. However they suffice to show a subsequence of (u_n) converges to a classical solution $u = u(\beta,K)$ of the modified equation. An additional argument shows $\|u_n\|_C \geq \gamma(\beta,K) > 0$ so u is a nontrivial solution of (1.8).

The above program will be carried out as a series of lemmas. First some brief preliminary remarks about N. From (1.5), we have if $v \in N$, $v = p(x+t) - p(-x+t) \equiv v^+ - v^-$ where $p \in L^2(S^1)$. Imposing the normalization

(2.1) $$[p] \equiv \int_0^{2\pi} p(s)ds = 0$$

makes p unique. A simple computation shows for $p,q \in L^2(S^1)$,

(2.2) $$\int_T p(x+t)\, q(-x+t)dxdt = \frac{1}{2}[p][q].$$

Hence for $v = v^+ - v^- \in N$, (2.2) and our normalization $[v^+] = [v^-] = 0$ imply

(2.3) $$\int_T v^2\, dxdt = \int_T ((v^+)^2 + (v^-)^2)dxdt\ .$$

With these observations in hand, we begin our estimates for $u_n = v_n + w_n \in N \oplus N^\perp$.

<u>Lemma 2.4:</u> If u_n is a critical point of $I|_{E_n}$ and $I(u_n) = c_n$, there are constants M_3, $M_4 > 0$

(2.5) $$\|v_n\|_C \leq \sqrt{\frac{2}{\pi}}\ \|v_{nt}\|_{L^2} \leq M_3/\beta$$

and

(2.6) $$\beta\|v_{ntt}\|_{L^2} \leq \|f_K(u_n)\|_{L^2} \leq M_4(1 + \|u_n^3\|_{L^2})$$

where M_3 is independent of n, β, and K and M_4 depends only on K.

<u>Proof:</u> From Lemma 1.13 we have

(2.7) $$I'(u_n)\varphi = 0$$

for all $\varphi \in E_n$. Choosing $\varphi = v_n \equiv v_n^+ - v_n^- \equiv p_n(x+t) - p_n(-x+t)$ yields

(2.8) $$\beta\|v_{nt}\|^2_{L^2} = \int_T f_K(u_n)\, v_n\, dxdt \leq \|v_n\|_C\, \|f_K(u_n)\|_{L^1}$$
$$\leq 2\|v_n^+\|_C\, \|f_K(u_n)\|_{L^1}$$

since $\|v_n^+\|_C = \|v_n^-\|_C$. Using the normalization $[p_n] = 0$, we also have

(2.9) $$\|v_n^+\|_C = \|p_n\|_{C(S^1)} \leq (2\pi)^{\frac{1}{2}}\, \|p_n'\|_{L^2(S^1)}\ .$$

Combining (2.9) with (2.8), (2.2)-(2.3), and (1.34) gives (2.5).

To obtain (2.6), let $\varphi = v_{ntt} \in E_n$ in (2.8). Then

$$(2.10) \qquad \beta \| v_{ntt} \|^2_{L^2} = -\int_T f_K(u_n) v_{ntt} dxdt \leq \| f_K(u_n) \|_{L^2} \| v_{ntt} \|_{L^2} .$$

The definition of $f_K(z)$ implies there is a constant $a = a(K)$ such that

$$|f_K(z)| \leq a(K)(1 + |z|^3)$$

with this observation, (2.10) implies (2.6).

Remark 2.11: Note that (2.10) contains a $\| f_K(u_n) \|_{L^2}$ term on its left hand side while Lemma 1.16 only provides us with a bound on $\| f_K(u_n) u_n \|_{L^1}$. The same estimates would have been obtained had we been working with f instead of f_K. It was to bridge this regularity gap that led us to introduce f_K. How this is done will be seen shortly in Lemma 2.18.

To continue, a representation theorem for solutions of

$$(2.12) \qquad \begin{cases} \square w = g, & 0 < x < \pi, \quad t \in \mathbb{R} \\ w(0,t) = 0 = w(\pi,t), & w(x,t+2\pi) = w(x,t) \end{cases}$$

is required.

Lemma 2.13: If $g \in C^j \cap N^\perp (H^j \cap N^\perp)$, $j \geq 0$, there exists a unique $w \in C^{j+1} \cap N^\perp$ $(H^{j+1} \cap \overset{\circ}{H}_1 \cap N^\perp)$ satisfying (2.12) and the map from $g \to w$ is continuous between these spaces. Moreover letting

$$(2.14) \qquad (\Psi g)(x,t) = -\frac{1}{2} \int_x^\pi \int_{t+x-\xi}^{t-x+\xi} g(\xi,\tau) d\tau d\xi + \frac{1}{2} \frac{\pi - x}{\pi} \int_0^\pi \int_{t-\xi}^{t+\xi} g(\xi,\tau) d\tau d\xi ,$$

for the C^j case, w has the representation

$$(2.15) \qquad w(x,t) = (\Psi g)(x,t) - Z\, Q((\Psi g)(x,t))$$

where

$$(Zp)(x,t) = p(x+t) - p(-x+t)$$

and

$$(Q\psi)(y) = \frac{1}{2\pi} \int_0^\pi (\psi(y-s,s) - \psi(y+s,s)) ds.$$

Remark 2.16 A proof of the C^j case can be found in [9] and the H^j case in [10]. For $j = 0$, the solution is only a weak solution of (2.12).

Some further notation is required at this point. For $\Omega \subset \mathbb{R}^n$, $\alpha \in (0,1)$, and $u \in C(\Omega,\mathbb{R})$, let

$$\|u\|_{C^\alpha(\Omega)} = \|u\|_{C(\Omega)} + \sup_{x \neq y \in \Omega} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \equiv \|u\|_{C(\Omega)} + H_\alpha(u) .$$

It is easy to verify that

$$\text{(2.17)} \qquad \begin{cases} \|Zp\|_{C^\alpha(T)} \leq 2\|p\|_{C^\alpha(S^1)} \\ \|Q\psi\|_{C^\alpha(S^1)} \leq 2\|\psi\|_{C^\alpha(T)} \end{cases} .$$

Returning to our estimates for u_n, we have

<u>Lemma 2.18:</u> There is a constant A_3 depending on K and β but independent of n such that

$$\text{(2.19)} \qquad \|w_n\|_{C^{\frac{1}{2}}} \leq A_3 .$$

<u>Proof:</u> Equation (2.7) implies

$$\text{(2.20)} \qquad \Box w_n = \beta v_{ntt} - P_n f_K(u_n) \equiv \zeta_n$$

where P_n denotes the orthogonal projector of $L^2(T)$ onto E_n. Applying Lemma 2.3 and (2.17) to (2.20) shows for any $\alpha \in (0,1)$,

$$\text{(2.21)} \qquad \|w_n\|_{C^\alpha} \leq \frac{5}{2} \|\Psi \zeta_n\|_{C^\alpha} .$$

To further estimate the right hand side of (2.21), let

$$\chi(x,t) = \int_x^\pi \int_{t+x-\xi}^{t-x+\xi} \zeta(\xi,\tau)\, d\tau d\xi .$$

Then by Schwarz' inequality,

$$\text{(2.22)} \qquad |\chi(x,t)| \leq \sqrt{2}\, \pi \|\zeta\|_{L^2}$$

and similarly

$$\text{(2.23)} \qquad |\chi(x+h,t+k) - \chi(x,t)| \leq \left(\int_A \zeta^2 d\tau d\xi\right)^{\frac{1}{2}} \left(\int_A \zeta^2 d\tau d\xi\right)^{\frac{1}{2}} \leq \alpha_1(|h| + |k|)^{\frac{1}{2}} \|\zeta\|_{L^2(T)}$$

where A denotes the region of integration for the difference in the two values of χ. Since $C^\alpha(T)$ is a Banach algebra,

$$\text{(2.24)} \qquad \|\varphi\chi\|_{C^\alpha} \leq \alpha_2 \|\varphi\|_{C^\alpha} \|\chi\|_{C^\alpha}$$

for all $\varphi, \chi \in C^{\alpha}$. Hence taking $\alpha = \frac{1}{2}$ and combining (2.20)-(2.24) yields

(2.25) $$\|w_n\|_{C^{\frac{1}{2}}} \le \alpha_3 \|\beta v_{ntt} - P_n f_K(u_n)\|_{L^2} \le \alpha_3(\beta\|v_{ntt}\|_{L^2} + \|f_K(u_n\|_{L^2}) .$$

Applying (2.6) we further conclude

(2.26) $$\|w_n\|_{C^{\frac{1}{2}}} \le \alpha_4(1 + \|u_n^3\|_{L^2})$$

where α_4 depends on K .

To improve (2.26) and obtain (2.19), some simple observations and an interpolation argument are used. By (1.36) and (2.5),

(2.27) $$\|u_n^3\|_{L^2} \le (\|v_n\|_C + \|w_n\|_C)\|u_n\|^2_{L^4} \le (\frac{M_3}{\beta} + \|w_n\|_C)A_2^2 .$$

Using the Hölder inequality and (1.36) and (2.5) again shows

(2.28) $$\|w_n\|_{L^4} = \|u_n - v_n\|_{L^4} \le \alpha_5$$

where α_5 depends on β and K .

By a general interpolation inequality [11], for $-\infty < \lambda \le \mu \le \nu < \infty$ and e.g. $\varphi \in C^{\infty}(T)$,

(2.29) $$\|\varphi\|_{L^{1/\mu}} \le \gamma_1 \|\varphi\|^{\frac{\nu-\mu}{\nu-\lambda}}_{L^{1/\lambda}} \|\varphi\|^{\frac{\mu-\lambda}{\nu-\lambda}}_{L^{1/\nu}}$$

where for $s < 0$, if ρ is the greatest integer in $-\frac{2}{s}$ and $-\alpha = \rho + \frac{2}{s}$,

$$\|\varphi\|_{L^s} = \begin{cases} \sum_{|\sigma|=\rho} \|D^{\sigma}\varphi\|_C & \text{if } \alpha = 0 \\ \sum_{|\sigma|=\rho} H_{\alpha}(D^{\sigma}\varphi) & \text{if } \alpha > 0 . \end{cases}$$

(Actually the inequality in [11] requires φ to have compact support but because of our boundary conditions in x and periodicity in t , the result readily extends to our case.) Choosing $\lambda = -\frac{1}{4}$ shows $\|\varphi\|_{L^{-4}} = H_{\frac{1}{2}}(\varphi)$. Further setting $\nu = \frac{1}{4}$ and $\mu = 0$ in (2.29) gives

(2.30) $$\|\varphi\|_C \le \gamma_1(H_{\frac{1}{2}}(\varphi))^{\frac{1}{2}} \|\varphi\|^{\frac{1}{2}}_{L^4} \le \gamma_1 \|\varphi\|^{\frac{1}{2}}_{C^{\frac{1}{2}}} \|\varphi\|^{\frac{1}{2}}_{L^4} .$$

Combining (2.26)-(2.28) and (2.30) then yields

(2.31) $$\|w_n\|_{C^{\frac{1}{2}}} \le \gamma_2(1 + \|w_n\|^{\frac{1}{2}}_{C^{\frac{1}{2}}}$$

from which the lemma follows.

Now that (2.19) has been established, it is easy to use bootstrap arguments to estimate higher derivative norms of (u_n).

Lemma 2.32: There is a constant A_4 depending on K and β such that

$$\|v_n\|_{H^3} + \|w_n\|_{H^2} \leq A_4 . \tag{2.33}$$

Proof: By Lemma 2.13 with $j = 1$, (2.5)-(2.6), and (2.19), we have

$$\|w_n\|_{H^1} \leq \alpha_1 \|\beta v_{ntt} - P_n f_K(u_n)\|_{L^2} \leq \alpha_2 . \tag{2.34}$$

From (2.7) with $\varphi = v_{nt^4}$, we obtain

$$\beta \|v_{nt^3}\|_{L^2}^2 \leq \left| \int_T f_K'(u_n) u_{nt} v_{nt^3} \, dxdt \right| \tag{2.35}$$

so (2.5), (2.19), (2.34) and the Schwarz inequality give

$$\|v_{nt^3}\|_{L^2} \leq \alpha_3 . \tag{2.36}$$

For $v \in N \cap C^k$, $k \geq 2$, $\Box v = 0$, and it is easy to verify that all derivatives of order k of v have the same L^2 norm. Hence (2.36) and (2.5)-(2.6) imply

$$\|v_n\|_{H^3} \leq \alpha_4 . \tag{2.37}$$

Finally Lemma 2.13 again with $j = 1$ and our above estimates show

$$\|w_n\|_{H^2} \leq \alpha_5 . \tag{2.38}$$

Thus (2.33) obtains

We can now solve (1.8).

Theorem 2.39: The modified equation (1.8) possesses a nontrivial classical solution.

Proof: Lemma 2.32 and standard embedding theorems imply that along a subsequence of the u_n, v_n converges in $C^2 \cap N$ to $v = v(\beta, K) \in H^3 \cap N$ and w_n converges in $H^2 \cap N^\perp$ (and therefore in $C^{\frac{1}{2}} \cap N^\perp$) to $w = w(\beta, K) \in H^2 \cap N^2$. Thu- we can pass to a limit in (2.7) to conclude for $u = v + w$ that

$$I'(u)\varphi = 0 \qquad \text{for all } \varphi \in \bigcup_{n \in \mathbb{N}} E_n . \tag{2.40}$$

This implies u satisfies (1.8) a.e. Since $f_K(u) \in C$ and w satisfies (2.15), we see that $w \in C^1$. Assuming for the moment that $v \in C^3$, Lemma 2.13 shows $w \in C^2$ and hence u is a classical solution of the modified equation.

To verify that $v \in C^2$, we write $v = p(x+t) - p(-x+t)$ with $[p] = 0$. From (2.40) or (1.6) it follows that

$$\int_T [-\beta(p''(x+t) - p''(-x+t)) + f_K(u)](\zeta(x+t) - \zeta(-x+t))dxdt = 0 \tag{2.41}$$

for all $\zeta \in L^2(S')$. Denoting the expression in brackets by $\psi(x,t)$ and using the periodicity of ζ and ψ in t shows

$$\int_0^\pi \int_0^{2\pi} \psi(x,t)\,\zeta(x+t)dxdt = \int_0^\pi \int_x^{2\pi-x} \psi(x,t)\,\zeta(x+t)dxdt = \int_0^\pi \int_0^{2\pi} \psi(x,s-x)\zeta(s)dxds \tag{2.42}$$

and

$$\int_0^\pi \int_0^{2\pi} \psi(x,t)\zeta(-x+t)dxdt = \int_0^\pi \int_0^{2\pi} \psi(x,s+x)\zeta(s)dxds. \tag{2.43}$$

Thus

$$\int_0^{2\pi} \zeta(s)[\int_0^\pi (\psi(x,s-x) - \psi(x,s+x))dx]\,ds = 0 \tag{2.44}$$

for all $\zeta \in L^2(S^1)$. Hence

$$\int_0^\pi (\psi(x,s-x) - \psi(x,s+x))dx = 0\,. \tag{2.45}$$

Substituting for ψ from (2.41) and simplifying yields

$$2\pi\,\beta\,p''(s) = \int_0^{2\pi} f_K(u(x,s-s)) - f_K(u(x,s+x))dx\,. \tag{2.46}$$

Since $w \in C'$ and $v \in C^2$, (2.46) shows p''' is continuous in s and therefore $v \in C^3$.

To complete the proof, we must show $u \not\equiv 0$, i.e. u is a nontrivial solution of (1.8). This is a consequence of the following lemma.

<u>Lemma 2.47:</u> For each $\beta, K > 0$, there is a constant $\gamma = \gamma(\beta, K) > 0$ independent of n such that

$$\|u_n\|_C \geq \gamma\,. \tag{2.48}$$

<u>Proof:</u> From (2.5), (2.8), we have

$$\beta\|v_n\|_C \leq \alpha_1 \|f_K(u_n)\|_C\,. \tag{2.49}$$

By $(f_3(i)$, for all $\varepsilon > 0$, there is a constant $A_\varepsilon = A_\varepsilon(K) > 0$ such that

$$z\,f_K(z) \leq \varepsilon\,|z|^2 + A_\varepsilon\,|z|^4 \tag{2.50}$$

for all $z \in \mathbb{R}$. Employing (2.50) in (2.49) shows

$$\|v_n\|_C \leq \frac{\alpha_1 \varepsilon}{\beta} \Lambda_n + \frac{\alpha_2}{\beta} \Lambda_n^3 \tag{2.51}$$

where $\Lambda_n = \|v_n\|_C + \|w_n\|_C$. From Lemma 2.13, (2.6), and (2.50) we get

(2.52) $$\|w_n\|_C \leq \alpha_3 \| -\beta v_{ntt} + P_n f_K(u_n)\|_{L^2} \leq 2\alpha_3 \|f_K(u_n)\|_{L^2} \leq \alpha_4(\varphi \Lambda_n + A_\varepsilon \Lambda_n^3).$$

Adding (2.51)-(2.52) yields

(2.53) $$\Lambda_n \leq (\frac{\alpha_1}{\beta} + 2\alpha_4)\varepsilon \Lambda_n + \alpha_5 \Lambda_n^3 .$$

Since the constants α_1 and α_4 are independent of n, on making $\varepsilon(\alpha_1\beta^{-1} + 2\alpha_4) \leq \frac{1}{2}$, we get

(2.54) $$\frac{1}{2}\Lambda_n \leq \alpha_5 \Lambda_n^3 \quad \text{or} \quad \Lambda_n \geq (2\alpha_5)^{-\frac{1}{2}} \equiv \gamma .$$

Thus the lemma and Theorem 2.39 are proved.

Remark 2.55: From (1.34) and the convergence of a subsequence of the u_n to $u(\beta,K)$ we get,

(2.56) $$\|f_K(u)\|_{L^1} \leq M_2 .$$

Remark 2.57: Had we only assumed (f_1'), (f_2'), and (f_3) (see Remark 1.37), the results of this section remain unchanged until Lemma 2.32 which is lost. However the uniform bounds obtained for $\|v_n\|_{H^2} + \|w_n\|_{C^{\frac{1}{2}}}$ are sufficient to get a subsequence of $v_n + w_n$ to converge to a function $u = v+w \in C$ which satisfies

(2.58) $$\int_T (u \Box \varphi - \beta v \varphi_{tt} + f(u)\varphi)dxdt = 0$$

for all $\varphi \in C_0^\infty(T)$. Arguments in the proof of Theorem 2.39 further show $v \in C^2$ and $w \in C^1$. Lastly the proof of Lemma 2.47 is unaffected for this case so u is a nontrivial weak solution of (1.8).

§3. A weak solution of (1.1).

In this section, the solutions $u = u(\beta, K)$ of (1.8) will be used to get a weak solution of (1.1). Our program is to first get an upper bound for $\|u(\beta,K)\|_C$ independent of β and K. This permits us to choose K so large that $f_K(u) = f(u)$ for $u = u(\beta, K)$. With K thus fixed, we show $\{u(\beta,K)\}$ is equicontinuous in $C(T)$ from which it easily follows that (1.1) has a weak solution. An additional comparison argument shows the solution we find is nontrivial.

To begin we have:

Lemma 3.1: Let $v = v(\beta,K)$. Then with M_2 as in (2.56),

(3.2) $$\beta\|v_{tt}\|_{L^1} \le M_2/\pi .$$

Proof: Writing $v(x,t) = p(x+t) - p(-x+t)$, from (2.46) we have:

(3.3) $$2\pi\,\beta\int_0^{2\pi}|p''(s)|ds \le \int_0^{2\pi}\int_0^{\pi}(|f_K(u(x,s-x))| + |f_K(u(x,s+x))|)dxds.$$

Reversing the steps which led from (2.41) to (2.46) then shows:

(3.4) $$\pi\,\beta\|p''\|_{L^1(S^1)} \le \|f_K(u)\|_{L^1}$$

from which (3.2) follows via (2.56).

Corollary 3.5: There is a constant M_5 independent of β and K such that

(3.6) $$\|w(\beta,K)\|_C \le M_5 .$$

Proof: From (2.15) we get

$$\|w\|_C \le \alpha_1\|-\beta v_{tt} + f_K(u)\|_{L^1} .$$

Hence the result follows from (3.2) and (2.56).

Our next goal is to get a bound for $\|v\|_C$ independent of β and K. This is somewhat delicate since we have to get a pointwise estimate while working with the projection of (1.8) on N.

Lemma 3.7: There is a constant A_5 such that for all β, $K > 0$,

(3.8) $$\|v(\beta,K)\|_C \le A_5 .$$

Proof: If $v \equiv 0$ for some β, K, we trivially have a bound. Hence we assume $v \not\equiv 0$ for what follows. From (2.41) we obtain

(3.9) $$\int_T (-\beta v_{tt} + f_K(u))\varphi \, dxdt = 0$$

for all $\varphi \in N$. Further assume $\varphi \in H^1$. Then (3.9) can be rewritten as

(3.10) $$\int_T [\beta v_t \varphi_t + (f_K(v+w) - f_K(w))\varphi] dxdt = -\int_T f_K(w)\varphi dxdt .$$

We will get the estimate (3.8) by choosing φ to be an appropriate nonlinear function of v.

Define a function $q : \mathbb{R} \to \mathbb{R}$ by $q(s) = 0$ if $|s| \leq M$; $q(s) = s - M$ if $s > M$; and $q(s) = s + M$ if $s < -M$. Further set $v \equiv p(x+t) - p(-x+t) \equiv v^+ - v^-$ with $[p] = 0$, $q^+ = q(v^+)$, $q^- = q(v^-)$, and choose $\varphi = q^+ - q^-$. Then $\varphi \in N$ by construction. Consider

(3.11) $$\int_T v_t \varphi_t \, dxdt = \int_T [q'(v^+)(v_t^+)^2 - q'(v^+)v_t^+ v_t^- - q'(v^-)v_t^- v_t^+ + q'(v^-)(v_t^-)^2] dxdt .$$

Since $[v_t^{\pm}] = 0$, the middle two terms on the right in (3.11) vanish by (2.2). The remaining terms are nonnegative since $q' \geq 0$. Hence (3.10) implies

(3.12) $$\int_T (f_K(v+w) - f_K(w))(q^+ - q^-) dxdt \leq \| f_K(w) \|_C \int_T (|q^+| + |q^-|) dxdt .$$

For any $\delta \geq 0$, let $T_\delta = \{(x,t) \in T \,|\, |v(x,t)| \geq \delta\}$, $T_\delta^+ = \{(x,t) \in T \,|\, v(x,t) > \delta\}$, and $T_\delta^- = T_\delta \setminus T_\delta^-$. By (f_2), the integrand on the left hand side of (3.12) is nonnegative. Therefore

(3.13) $$\int_T (f_K(v+w) - f_K(w))(q^+ - q^-) dxdt \geq \int_{T_\delta} (f_K(v+w) - f_K(w)) \cdot (q^+ - q^-) \, dxdt.$$

Define

(3.14) $$\psi_K(z) = \begin{cases} \min\limits_{|\zeta| \leq M_5} f_K(z+\zeta) - f_K(\zeta), & z \geq 0 \\ \max\limits_{|\zeta| \leq M_5} f_K(z+\zeta) - f_K(\zeta), & z < 0 \end{cases}$$

with M_5 from (3.6). By (f_2), $\psi_K(z)$ is strictly monotonically increasing, and by (1.7), $\psi_K(z) \to \pm\infty$ as $z \to \pm\infty$. By (3.14),

(3.15) $$\int_{T_\delta^+} (f_K(v+w) - f_K(w))(q^+ - q^-) dxdt \geq \int_{T_\delta^+} \frac{\psi_K(v)}{v} v(q^+ - q^-) dxdt$$

$$\geq \frac{\psi_K(\delta)}{\|v\|_C} \int_{T_\delta^+} v(q^+ - q^-) dxdt$$

since $v(q^+ - q^-) \geq 0$. Observing that for $z < 0$

(3.16) $$\psi_K(z) = -\min_{|\zeta| \leq M_5} f_K(\zeta) - f_K(z+\zeta)$$

we similarly find:

$$(3.17)\qquad \int_{T_{\delta}-} (f_K(v+w) - f_K(w))(q^+ - q^-)dxdt \ge \frac{\psi_K(-\delta)}{\|v\|_C} \int_{T_{\delta}-} v(q^+ - q^-)dxdt \ .$$

For $z \geq 0$, let $\mu_K(z) = \min(\psi_K(z), -\psi_K(-z))$. Then $\mu_K(z)$ is strictly monotonically increasing and $\mu_K(z) \to \infty$ as $z \to \infty$. By (3.15), (3.17) we have

$$(3.18)\qquad \int_{T_\delta} (f_K(v+w) - f_K(w))(q^+ - q^-)dxdt \ge \frac{\mu_K(\delta)}{\|v\|_C} \int_{T_\delta} v(q^+ - q^-)dxdt$$

$$= \frac{\mu_K(\delta)}{\|v\|_C} \left[\int_T (v^+ - v^-)(q^+ - q^-)dxdt - \int_{T\setminus T_\delta} v(q^+ - q^-)dxdt\right]$$

$$\geq \frac{\mu_K(\delta)}{\|v\|_C} \left[\int_T (v^+q^+ + v^-q^-)dxdt - \delta \int_T (|q^+| + |q^-|)dxdt\right] ,$$

the last inequality following via (2.2) since $[v^\pm] = 0$.

The definition of $q(s)$ implies $s\, q(s) \geq M|q(s)|$. Hence (3.12), (3.13), and (3.18) imply:

$$(3.19)\qquad \|f_K(w)\|_C \int_T (|q^+| + |q^-|)dxdt \ge \frac{(M-\delta)\mu_K(\delta)}{\|v\|_C} \int_T (|q^+| + |q^-|)dxdt$$

Choosing any $M < \|v^\pm\|_C$, the integral term in (3.19) is positive so we can divide by it and obtain:

$$(3.20)\qquad \frac{M-\delta}{\|v\|_C} \mu_K(\delta) \le \|f_K(w)\|_C \ .$$

Since (3.20) is valid for any $M < \|v^\pm\|_C$, we can pass to a limit and let $M = \|v^\pm\|_C$. Further noting that $\|v\|_C \leq 2\|v^\pm\|_C$, (3.20) implies

$$(3.21)\qquad \frac{\|v^\pm\|_C - \delta}{2\|v^\pm\|_C} \mu_K(\delta) \le \|f_K(w)\|_C \ .$$

Choosing e.g. $\delta = \frac{1}{2}\|v^\pm\|_C$, we get

$$(3.22)\qquad \mu_K(\tfrac{1}{2}\|v^\pm\|_C) \le 4\|f_K(w)\|_C \ .$$

By (3.6), the right hand side of (3.22) is bounded independently of β and K. By (f_3)(ii) and (1.7), given any $\Lambda > 0$, there is a $z_0(\Lambda)$ such that $|f_K(z)| \geq \Lambda$ for all $|z| \geq z_0$ and for all K. Using the definition of μ_K, (3.8) follows from these observations and the lemma is proved.

Remark 3.23: Related but simpler arguments using q can be found in [1], [10]. A somewhat cruder argument using (f_3)(ii) and avoiding (f_2) could have been employed to obtain (3.8). However the above proof makes the equicontinuely argument of Lemma 3.29 much briefer.

Remark 3.24: Henceforth we take $K = M_5 + A_5$ so at a solution of (1.8) we have $f_K(u) = f(u)$. Thus we can and will generally suppress K in what follows.

Corollary 3.25: There is a constant M_6 independent of β such that

$$\|w\|_{C^1} \leq M_6 . \tag{3.26}$$

Proof: By Lemma 2.13,

$$\|w\|_{C^1} \leq \alpha_1 \|-\beta v_{tt} + f(u)\|_C . \tag{3.27}$$

From (2.46) we get

$$\beta \|v_{tt}\|_C \leq 2\|f(u)\|_C . \tag{3.28}$$

Hence (3.26) follows from (3.27)-(3.28), (3.6), and (3.8).

The last preliminary needed to obtain a weak solution of (1.1) is

Lemma 3.29: The functions $v = v(\beta)$ form an equicontinuous family in $C \cap N$.

Proof: The proof is similar to Lemma 3.7 so we will be brief. Let $u = v + w$ be a solution of (1.8) and $h \in \mathbb{R}$. Set $\hat{v}(x,t) = v(x,t+h)$, $\hat{w}(x,t) = w(x,t+h)$, $\hat{u} = \hat{v} + \hat{w}$, $V = \hat{v} - v$, $W = \hat{w} - w$, and $U = V + W$. From (2.41) we have

$$\int_T (-\beta V_{tt} + f(\hat{u}) - f(u))\, \varphi \, dxdt = 0 \tag{3.30}$$

for all $\varphi \in N$. This can also be written as

$$\int_T \beta V_t \varphi_t \, dxdt + \int_T (f(\hat{v}+w) - f(u))\, \varphi \, dxdt = -\int_T (f(\hat{u}) - f(\hat{v}+w))\varphi \, dxdt . \tag{3.31}$$

Choosing $\varphi = q(V^+) - q(V^-) \equiv Q^+ - Q^-$ where $V^+ = \hat{v}^+ - v^+$, etc. yields the analogue of (3.12):

$$\int_T (f(V+u) - f(u))(Q^+ - Q^-)dxdt \leq \|f(\hat{u}) - f(\hat{v}+w)\|_C \int_T (|Q^+| + |Q^-|)dxdt. \tag{3.32}$$

From (3.6), (3.8), and (3.26) we see

$$\|f(\hat{u}) - f(\hat{v}+w)\|_C \leq \gamma_1 \|W\|_C \leq \gamma_1 M_6 |h| . \tag{3.33}$$

Next let $\psi(z)$, $\mu(z)$ be as in Lemma 3.7 where we drop the subscript K and replace M_5 by $M_5 + A_5$. Arguing as in Lemma 3.7 we find

(3.34) $$\int_T (f(V+u) - f(u))(Q^+ - Q^-)dxdt \geq \frac{\mu(\delta)}{\|V\|_C}(M-\delta)\int_T(|Q^+| + |Q^-|)dxdt$$

and

(3.35) $$\mu(\tfrac{1}{2}\|V^{\pm}\|_C) \leq 4\gamma_1 M_6 |h|$$

or

(3.36) $$\max_{s \in S^1} |p(s+h) - p(s)| \leq 2\mu^{-1}(4\gamma_1 M_6 |h|)$$

where as usual $v(x,t) = p(x+t) - p(-x+t)$. Thus (3.36) provides us with a modulus of continuity for v independent of $\pm$ and the lemma is proved.

Theorem 3.37: If f satisfies (f_1)-(f_3), (1.1) possesses a nontrivial weak solution $u = v+w \in (C \cap N) \oplus (C^1 \cap N^{\perp})$ satisfying

(3.38) $$\int_T (u \Box \varphi + f(u)\varphi)dxdt = 0$$

for all $\varphi \in C_0^{\infty}(T)$.

Proof: Let $u(\beta) = v(\beta) + w(\beta)$ denote a family of solutions of (1.8) for $\beta > 0$. By Corollary 3.25, the functions $w(\beta)$ are bounded in $C^1 \cap N^{\perp}$ and by Lemma 3.7 and 3.29 the functions $v(\beta)$ are uniformly bounded and equicontinuous in $C \cap N$. Hence as $\beta \to 0$ along some subsequence, $u(\beta) \to v+w \in (C \cap N) \oplus (C \cap N^{\perp})$. Thus writing (1.8) in its weak form and passing to a limit gives (3.38).

To see that $w \in C^1 \cap N^{\perp}$, it suffices to show that w satisfies (2.15) with $g = -f(u)$. By (3.8), (3.28) and an interpolation inequality [11],

(3.39) $$\beta\|v_{tt}(\beta)\|_{L^1} \leq \alpha_1 \beta\|v_{tt}(\beta)\|_C^{\frac{1}{2}} \|v(\beta)\|_C^{\frac{1}{2}} \leq \alpha_1\beta(\frac{2}{\beta} f(A_5 + M_5))^{\frac{1}{2}} A_5^{\frac{1}{2}} \to 0$$

as $\beta \to 0$. Since $\beta\|v_{tt}\|_{L^1} \to 0$, we can pass to a limit in (2.15) for $w(\beta)$ to get (2.15) for w.

It remains to show u is a nontrivial solution of (1.1). This is a more difficult problem than that confronted in Lemma 2.47 since we no longer can use the β term to help us as earlier and (f_3)(i) gives us no information on how rapidly $z^{-1}f(z) \to 0$ as $z \to 0$. We get around these difficulties with the aid of a comparison argument. The following lemma completes the proof of Theorem 3.37.

Lemma 3.40: There is a $\gamma > 0$ such that $\|w(\beta)\|_C \geq \gamma$ for all β near 0.

Proof: From (2.50) we see

$$F_K(z) \le \frac{\varepsilon}{2} z^2 + \frac{A_\varepsilon}{4} z^4 \tag{3.41}$$

for all $z \in \mathbb{R}$. Hence

$$I(u) \ge J(u) \equiv \int_T [\tfrac{1}{2}(u_t^2 - u_x^2 - \beta v_t^2 - \varepsilon u^2) - \frac{A_\varepsilon}{4} u^4] dxdt \tag{3.42}$$

for all $u \in \bigcup_{n \in \mathbb{N}} E_n$. Recalling that

$$\Gamma_n = \{h \in C(\bar{B}_{R(n)} \cap \mathbb{R}^{k(n)+1}, E_n) \mid h(u) = u \quad \text{if} \quad I(u) \le 0\},$$

we have

$$c_n = \inf_{h \in \Gamma_n} \max_{u \in \bar{B}_R \cap \mathbb{R}^{k+1}} I(h(u)) \ge \inf_{h \in \Gamma_n} \max_{u \in \bar{B}_R \cap \mathbb{R}^{k+1}} J(h(u)) \equiv b_n . \tag{3.43}$$

We claim that b_n is a critical value of $J|_{E_n}$. Indeed for $u \in N^+ \cap E_n \equiv (\mathbb{R}^k)^\perp$,

$$\frac{1}{2}\int_T (u_t^2 - u_x^2) dxdt \ge \frac{3}{2}\int_T u^2 \, dxdt. \tag{3.44}$$

Since for $u \in E_n$, $\|u\|^4_{L^4} = o(\|u\|^2_{L^2})$ at $u = 0$, by requiring that $\varepsilon < 3$, we see there exists an $r = r(n, K) > 0$ such that $J(u) > 0$ for $u \in (B_r \setminus \{0\}) \cap (\mathbb{R}^k)^\perp$. It is now easily verified that J satisfies the remaining hypotheses of Theorem 1.10 and b_n as defined in (3.43) is a critical value of $J|_{E_n}$ with $b_n > 0$. From (3.43) and (1.28), we have

$$c_n = \int_T [\tfrac{1}{2} f_K(u_n) u_n - F_K(u_n)] dxdt \ge b_n = \tfrac{1}{4} A_\varepsilon \int_T \bar{u}_n^4 \, dxdt \tag{3.45}$$

where $\bar{u}_n(\beta, K)$ is a critical point of $J|_{E_n}$ corresponding to the critical value b_n. As was observed in Remark 1.15, the fact that f satisfies (f_3)(i) was only used to verify the hypotheses of Theorem 1.10 and would not be used again until Lemma 2.47. Since J does satisfy the hypo-hypotheses of Theorem 1.10 and $\varepsilon z + A_\varepsilon z^3$ satisfies (f_1), (f_2), and (f_3)(ii), it follows from the estimates of §1-2 that along a subsequence of $n \to \infty$, $\bar{u}_n(\beta, K)$ converges to a classical solution $\bar{u}(\beta, K)$ of

$$\begin{aligned} &\Box \bar{u} - \beta \bar{v}_{tt} + \varepsilon \bar{u} + A_\varepsilon \bar{u}^3 = 0 \\ &\bar{u}(0,t) = 0 = \bar{u}(\pi, t) \; ; \; \bar{u}(x, t+2\pi) = \bar{u}(x,t) \end{aligned} \tag{3.46}$$

Moreover from (3.45), we have

$$\int_T [\tfrac{1}{2} f_K(u(\beta, K)) - F_K(u(\beta, K))] dxdt \ge \tfrac{1}{4} A_\varepsilon \int_T \bar{u}(\beta, K)^4 dxdt . \tag{3.47}$$

To see that $\bar{u}$ is a nontrivial solution of (3.46) requires a slight modification of the proof of Lemma 2.47. Since

$$(3.48) \qquad J'(\bar{u}_n)\bar{v}_n = 0 ,$$

we find that

$$(3.49) \qquad \beta\|\bar{v}_{n_t}\|^2_{L^2} + \varepsilon\|\bar{v}_n\|^2_{L^2} \le A_\varepsilon\|\bar{v}_n\|_C \int_T |\bar{u}_n|^3 dxdt .$$

Using (3.49), we can replace (2.49) by

$$(3.50) \qquad \beta\|\bar{v}_n\|_C \le \alpha_1 A_\varepsilon \|\bar{u}_n^3\|_C \le \alpha_1 A_\varepsilon \bar{\Lambda}_n^3$$

with $\bar{\Lambda}_n = \|\bar{v}_n\|_C + \|\bar{w}_n\|_C$. Combining (3.50) with (2.52) which remains valid for $\bar{u}_n$, shows

$$(3.51) \qquad \beta\,\bar{\Lambda}_n \le \alpha_1 A_\varepsilon \bar{\Lambda}_n^3 + \alpha_4\,\beta(\varepsilon\,\bar{\Lambda}_n + A_\varepsilon\,\bar{\Lambda}_n^3)$$

where α_4 is independent of ε, n, β, K. Choosing $\varepsilon < (2\alpha_4)^{-1}$ given $\bar{\Lambda}_n \ge \beta^{\frac{1}{2}} A_\varepsilon^{-\frac{1}{2}}(\alpha_1 + \alpha_4)^{-\frac{1}{2}}$. Hence $\|\bar{u}(\beta,K)\|_C$ satisfies the same inequality.

Now to obtain the lower bound for $\|w(\beta)\|_C$ (where we suppress the dependence on K via Remark 3.24) suppose to the contrary that $w(\beta) \to 0$ along some subsequence. From (3.22) we conclude the same is true for $v(\beta)$ and hence $u(\beta)$. Then by (3.47), $\|\bar{u}(\beta)\|_{L^4} \to 0$ as $\beta \to 0$. Applying (3.4) for $\bar{u}$ and (2.15) shows

$$(3.52) \qquad \|\bar{w}(\beta)\|_C \le \alpha_5 \|\varepsilon\,\bar{u}(\beta) + A_\varepsilon\,\bar{u}(\beta)^3\|_{L^1} \to 0$$

as $\beta \to 0$. Next employing (3.10) for the $\bar{u}$ problem with $\varphi = \bar{v}$ gives

$$(3.53) \qquad \int_T (\bar{v}^4 + 3\,\bar{v}^3\,\bar{w} + 3\,\bar{v}^2\,\bar{w}^2)dxdt = -\int_T \bar{v}\,\bar{w}^3\,dxdt.$$

An application of the Hölder inequality implies

$$(3.54) \qquad \|\bar{v}(\beta)\|_{L^4} \le \alpha_6\|\bar{w}(\beta)\|_{L^4} .$$

Since $\bar{v} \equiv \bar{v}^+ - \bar{v}^-$ with $[\bar{v}^\pm] = 0$ and by (2.2),

$$(3.55) \qquad \int_T \bar{v}^4 dxdt = \int_T ((\bar{v}^+)^4 + 6(\bar{v}^+)^2(\bar{v}^-)^2 + (\bar{v}^-)^4)dxdt ,$$

we have

$$(3.56) \qquad \int_T ((\bar{v}^+)^4 + (\bar{v}^-)^4)dxdt \le \alpha_6^+ \|\bar{w}\|^4_{L^4} .$$

Returning to (3.10) with $\varphi = q(\bar{v}^+) - q(\bar{v}^-) \equiv q^+ - q^-$ and making some crude estimates gives:

$$(3.57) \qquad \int_T \bar{v}^3(q^+ - q^-)dxdt \le \alpha_7(\|\bar{v}^\pm\|_C^2 + \|\bar{w}\|_C^2)\|\bar{w}\|_C \int_T (|q^+| + |q^-|dxdt .$$

Expanding the left integral and using (2. 2) again shows:

$$(3.58) \qquad \int_T \bar{v}^3(q^+-q^-)dxdt \geq \int_T[(\bar{v}^+)^3 q^+ + (\bar{v}^-)^3 q^- - (\bar{v}^+)^3 q^- - (\bar{v}^-)^3 q^+]dxdt$$

where we dropped two nonnegative terms from the right hand side. Using (2. 2), (3. 56), and the Hölder inequality to estimate the last two terms on the right in (3. 58) leads to

$$(3.59) \qquad M^3 \int_T (|q^+| + |q^-|)dxdt \leq \int_T[(\bar{v}^+)^3 q^+ + (\bar{v}^-)^3 q^-]dxdt$$

$$\leq \alpha_8(\|\bar{v}^{\pm}\|_C^2 + \|\bar{w}\|_C^2)\|\bar{w}\|_C \int_T (|q^+| + |q^-|)dxdt$$

(since $s^3 q(s) \geq M^3|q(s)|$). Arguing as in Lemma 3.7, we conclude

$$(3.60) \qquad \|\bar{v}(\beta)\|_C \leq \alpha_9\|\bar{w}(\beta)\|_C \,.$$

Returning to (3. 52) again and using (3. 60) yields

$$(3.61) \qquad \|\bar{w}(\beta)\|_C \leq 2\pi^2 \alpha_5[\varepsilon(1+\alpha_9)\|\bar{w}(\beta)\|_C + A_\varepsilon(1+\alpha_9)^3\|\bar{w}(\beta)\|_C^3] \,.$$

Note that the constants α_5, α_9, and A_ε are independent of β and α_5 and α_9 are independent of ε. Moreover (3. 61) holds for all $0 < \varepsilon < \min(3,(2\alpha_4)^{-1})$. Thus further choosing $0 < \varepsilon \leq [4\pi^2 \alpha_5(1+\alpha_9)]^{-1}$ shows

$$\|\bar{w}(\beta)\|_C \geq (2\pi^2 \alpha_5 A_\varepsilon(1+\alpha_9)^3)^{-\frac{1}{2}}$$

contrary to (3. 52). Thus Lemma 3. 40 and Theorem 3. 37 are proved.

§4. Regularity of the weak solution.

We have shown (1.1) possesses a weak solution $u = v+w$ with $v \in C \cap N$ and $w \in C^1 \cap N^\perp$. In this section we shall prove that in fact $v \in C^1 \cap N$ and $w \in C^2 \cap N^\perp$. It then follows that

$$\Box w + f(u) = 0 , \qquad 0 < x < \pi , \qquad t \in \mathbb{R}$$

together with our boundary and periodicity conditions. This is not quite a classical solution of (1.1) since we only have $v \in C^1$. However assuming f has more derivatives, we shall show greater regularity obtains for u. This will complete the proof of Theorem 1.6. We are indebted to L. Nirenberg for some of the ideas used in the proof of the following result.

<u>Theorem 4.1:</u> If f satisfies (f_1)-(f_3) and $u = v+w$ is a nontrivial weak solution of (1.1) with $v \in C \cap N$, $w \in C^1 \cap N^\perp$, then $v \in C^1 \cap N$ and $w \in C^2 \cap N^\perp$.

<u>Proof:</u> First we show $v \in C^1$. Once that has been established, it follows from Lemma 2.13 that $w \in C^2$. Two cases are considered:

<u>Case 1:</u> Suppose there is an $\bar{s} \in [0, 2\pi]$ such that $u(x, \bar{s}-x) \equiv \alpha$, a constant, for all $x \in [0, \pi]$. Then $\alpha = 0$ via our boundary conditions. Writing $v(x,t) = p(x+t) - p(-x+t)$ with $[p] = 0$ and setting $t = s-x$, for $s = \bar{s}$ we have

$$p(\bar{s} - 2x) = p(\bar{s}) + w(x, \bar{s} - x), \qquad x \in [0, \pi] . \tag{4.2}$$

The right hand side of (4.2) is continuously differentiable with respect to x. Therefore so is the left hand side. Hence p and v are continuously differentiable. Moreover $\|p'\|_{C(S^1)} \leq \frac{1}{2}\|w\|_C$ so

$$\|v\|_{C^1} \leq \|w\|_{C^1} . \tag{4.3}$$

<u>Case 2:</u> Suppose there is no $\bar{s} \in [0, 2\pi]$ such that $u(x, \bar{s}-x) = 0$ for all $x \in [0, \pi]$. Then (f_2) implies there is a $\gamma > 0$ such that

$$\int_0^\pi f_u(u(x, s-x))dx \geq \gamma \tag{4.4}$$

for all $s \in [0, 2\pi]$. Since $u(\beta) \to u$ in C as $\beta \to 0$ along a subsequence, for all small such β we have:

$$\int_0^\pi f_u(u(\beta)(x, s-x))dx \geq \frac{\gamma}{2} > 0 \tag{4.5}$$

for all $s \in [0, 2\pi]$. Differentiating (2.46) with respect to s shows

$$(4.6) \qquad 2\pi\beta\, p'''(s) = \int_0^\pi \{f_u(u(x,s-x))[p'(s) - p'(s-2x) + \frac{\partial}{\partial s} w(x,s-x)] -$$
$$- f_u(u(x,s+x))[p'(s+2x) - p'(s) + \frac{\partial}{\partial s} w(x,s+x)]\}dx$$

where we have not explicitly noted the dependence of p and u on β. This equation can be rewritten as

$$(4.7) \qquad -2\pi\beta\, p'''(s) + p'(s)\int_0^\pi (f_u(u(x,s-x)) + f_u(u(x,s+x)))dx =$$
$$= \int_0^\pi [f_u(u(x,s-x))(p'(s-2x) - \frac{\partial}{\partial s} w(x,s-x)) + f_u(u(x,s+x))(p'(s+2x) + \frac{\partial}{\partial s} w(x,s+x)]dx.$$

Observe that

$$(4.8) \qquad \begin{cases} \dfrac{d}{dx} f(u(x,s-x)) = f_u(u(x,s-x))(2\,p'(s-2x) + \dfrac{\partial}{\partial x} w(x,s-x)) \\ \dfrac{d}{dx} f(u(x,s+x)) = f_u(u(x,s+x))(2\,p'(s+2x) + \dfrac{\partial}{\partial x} w(x,s+x)) . \end{cases}$$

Hence on integrating (4.8), using (f_1), and the boundary conditions for u, we find:

$$(4.9) \qquad \begin{cases} \int_0^\pi f_u(u(x,s-x))\, p'(s-2x)dx = -\frac{1}{2}\int_0^\pi f_u(u(x,s-x)) \dfrac{\partial}{\partial x} w(x,s-x)dx \\ \int_0^\pi f_u(u(x,s+x))\, p'(s+2x)dx = -\frac{1}{2}\int_0^\pi f_u(u(x,s+x)) \dfrac{\partial}{\partial x} w(x,s+x)dx . \end{cases}$$

Substitution of (4.9) into (4.7) yields

$$(4.10) \qquad -2\pi\beta\, p'''(s) + p'(s)\int_0^\pi (f_u(u(x,s-x)) + f_u(u(x,s+x))dx =$$
$$= \int_0^\pi \{f_u(u(x,s-x))[-\frac{1}{2}\frac{\partial}{\partial x} w(x,s-x) - \frac{\partial}{\partial s} w(x,s-x)]$$
$$+ f_u(u(x,s+x))[-\frac{1}{2}\frac{\partial}{\partial x} w(x,s+x) + \frac{\partial}{\partial s} w(x,s+x)]\}dx .$$

Hence $\varphi(s) = p'(s)$ satisfies an equation of the form

$$(4.11) \qquad -2\pi\beta\,\varphi''(s) + a(s)\,\varphi(s) = h(s)$$

where $h \in C(S^1)$ and is bounded independently of $\beta > 0$. Moreover $a(s) \geq \gamma/2$. Since p' must vanish somewhere in $[0,2\pi]$ and $p' \not\equiv 0$, φ has a positive maximum and a negative minimum. From (4.11) we then conclude

$$(4.12) \qquad \|\varphi\|_{C(S^1)} \leq \frac{2}{\gamma}\|h\|_{C(S^1)}$$

and

$$(4.13) \qquad \|p'(\beta)\|_{C(S^1)} \leq \frac{2}{\gamma}\alpha_1\|w(\beta)\|_{C^1} \leq \frac{2}{\gamma}\alpha_1 M_6 .$$

It follows that $p(s) = \lim p(\beta)(s)$ is in $H^1(S^1)$. Multiplying (4.10) by $\psi \in L^2(S^1)$, integrating over $[0, 2\pi]$ and letting $\beta \to 0$, we see that (4.10) is valid in an a.e. sense with $\beta = 0$. Then (4.4) and $v \in C$, $w \in C^1$ imply $p' \in C(S^1)$ so $v \in C^1$. The proof is complete.

Corollary 4.14: If f is k times continuously differentiable, $k \geq 1$, then $v \in C^k$ and $w \in C^{k+1}$.

Proof: The proof is by induction on k. It has already been established for $k = 1$. Assume it for $k = j-1$. To get the result for $k = j$, note first that $v \in C^j$ implies that $w \in C^{j+1}$, again via Lemma 2.13. To get $v \in C^j$, we consider the two cases of Theorem 4.1. In Case 1, (4.2) can be differentiated j times since $w \in C^j$ and this gives the result. In Case 2, consider (4.10) with $\beta = 0$. Dividing by

$$\int_0^{\pi}(f_u(u(x,s-x)) + f_u(u(x,s+x))dx$$

the resulting right hand side is $j-1$ times continuously differentiable via (4.4) and the induction hypothesis. Hence $p' \in C^{j-1}$ and $v \in C^j$.

Remark 4.15: It is worth observing at this point that the arguments of §2-4 show that if $u = v+w$ is a weak solution of (1.1) with

(4.16) $$\|f(u)\|_{L^1} < \infty$$

where f satisfies (f_1), (f_2) and

(f_3') $$|f(z)| \to \infty \quad \text{as} \quad |z| \to \infty,$$

then $v \in C^1$ and $w \in C^2$. Moreover if $f \in C^k$, $k \geq 1$, then $v \in C^k$ and $w \in C^{k+1}$.

§5. Various extensions.

This section contains extensions of Theorem 1.6 in four directions. First we prove the result stated in the introduction concerning the existence of nontrivial periodic solutions of (0.1) for any period which is a rational multiple of π. The next two generalizations involve weakening the hypotheses (f_1)-(f_3). If (f_1)-(f_2) are replaced by (f_1')-(f_2'), we can still get a "weak" solution of (1.1) (Theorem 5.6) while if $(f_3)(i)$ is weakened by adding a linear term with the proper sign to f, an analogue of Theorem 1.6 (Theorem 5.13) still obtains. Lastly we study the forced vibration question for (0.1), i.e. we permit f to depend on x and t in addition to u. Since the above results mainly involve minor modifications of the arguments of §1-4, we will generally be sketchy with details here.

__Theorem 5.1:__ Let f satisfy (f_1)-(f_3). Then for any j, $m \in \mathbb{N}$, the problem

$$(5.2) \qquad \begin{cases} u_{tt} - u_{xx} + f(u) = 0, & 0 < x < \pi, \quad t \in \mathbb{R}, \\ u(0,t) = 0 = u(\pi,t) & \end{cases}$$

possesses a nontrivial solution $u = v + w \in (C^1 \cap N) \oplus (C^2 \cap N^\perp)$ which has the period $\frac{j}{m}\pi$ in t.

__Proof:__ It suffices to prove the result for $j = 1$. Only small modifications need be made in §1-4. The most important changes are the following: We replace T by $T_m = \{(x,t) \in [0,\pi]\times[0,\frac{\pi}{m}]\}$; the spaces L^2, C^k, etc introduced in §1 now are taken relative to T_m; N is replaced by

$$\{p(x+t) - p(-x+t) \mid p \text{ is } \tfrac{\pi}{m} \text{ periodic and } \int_0^{\frac{\pi}{m}} p^2(s)ds < \infty\}$$

and E_n by span $\{\sin jx \sin 2mkt, \sin jx \cos 2mkt \mid 0 \le j,k \le n\}$. It is straightforward to verify that Lemma 2.13 is valid for this new class of functions with no change in the representation (2.15) but only in the underlying class of functions. The results of §1-4 now go through with minor changes and Theorem 5.1 obtains

__Corollary 5.5:__ Under the hypotheses of Theorem 5.1, (5.2) possesses infinitely many distinct nontrivial solutions which are periodic in t.

__Proof:__ By Theorem 5.1, for each $n \in \mathbb{N}$, (5.2) possesses a nontrivial solution having period $\frac{\pi}{n}$ in t. Hence infinitely many of these solutions must be distinct.

Next we study the effect of weakening (f_1)-(f_2) to (f_1')-(f_2')(See Remarks 1.37 and 2.57) on Theorem 1.6. Here as was pointed out to us by H. Brezis the monotonicity of f can be exploited to get a weak solution of (1.1) using a standard monotonicity argument.

<u>Theorem 5.6:</u> If f satisfies (f_1'), (f_2'),(f_3), then (3.38) possesses a nontrivial solution $u = v+w$ with $v \in L^\infty \cap N$ and $w \in C \cap N^\perp$.

<u>Proof:</u> Remark 2.57 gives us a nontrivial solution $u(\beta,K)$ of (2.58) with $v(\beta,K) \in C^2$ and $w(\beta,K) \in C^1$. The argument of Corollary 3.5 goes unchanged for this case to give a bound on $\|w(\beta,K)\|_C$ independent of β and K. Since $\mu_K(z) \to \infty$ as $z \to \infty$, we still obtain (3.8). Hence we can drop the dependence of u on K as earlier. The bound (3.26) remains valid here. Unfortunately the proof of Lemma 3.29 no longer holds. Nevertheless one can still get the convergence of a subsequence of the $u(\beta)$ as $\beta \to 0$. Indeed since the functions $u(\beta)$ are bounded in $L^2 \cap L^\infty$, $\Box u(\beta)$ are bounded in L^2, $w(\beta)$ are bounded in C^1, and $\beta v_{tt} \to 0$ weakly in L^2 as $\beta \to 0$, it follows that we have

$$u(\beta) \xrightarrow{L^2} u, \quad \Box u(\beta) \xrightarrow{L^2} g, \quad w(\beta) \xrightarrow{C} w$$

along a subsequence of $\beta \to 0$. Note that if

$$h_n \xrightarrow{L^2} h \quad \text{and} \quad \Box h_n \xrightarrow{L^2} \zeta ,$$

then for all $\varphi \in C_0^\infty$,

$$\int_T \varphi\, \zeta \, dxdt = \lim_{n\to\infty} \int_T \varphi \, \Box h_n dxdt = \lim_{n\to\infty} \int_T h_n \Box \varphi \, dxdt = \int_T h \, \Box \varphi \, dxdt.$$

Thus $\Box h = \zeta$ weakly and $\Box h_n \xrightarrow{L^2} \Box h$. Hence $\Box$ has a closed graph as a mapping of $L^2_{weak} \to L^2_{weak}$ so $\Box u(\beta) \xrightarrow{L^2} \Box u$.

For any $\rho \in L^\infty$, by (f_2'),

(5.7) $$(f(u(\beta)) - f(\rho))(u(\beta) - \rho) \geq 0 .$$

Integrating over T and using (2.58) yields

(5.8) $$\int_T (\beta v_{tt}(\beta) - \Box u(\beta) - f(\rho))(u(\beta) - \rho) dxdt \geq 0 .$$

Letting $\beta \to 0$ along our subsequence in (5.8) and noting that

$$\beta \int_T v_{tt}(\beta)\, u(\beta)\, dxdt = -\beta \| v_t(\beta)\|^2_{L^2} \leq 0 ,$$

we get

$$(5.9) \qquad \int_T (-\Box u - f(\rho))\,(u-\rho)dxdt \geq 0$$

for all $\rho \in L^\infty$. Choose $\rho = u + \lambda\chi$, $\chi \in C^\infty$, $\lambda > 0$. Hence (5.9) becomes

$$(5.10) \qquad \int_T (-\Box u - f(u+\lambda\chi))(-\lambda\chi)dxdt \geq 0 .$$

Dividing by λ, letting $\lambda \to 0$, and noting that χ is arbitrary we find

$$\Box u + f(u) = 0 \quad \text{a.e.}$$

from which (3.38) and Theorem 5.6 follow.

Remark 5.11: Lemma 2.13 implies the first derivative of w is in L^∞.

For our next generalization of Theorem 1.6, we consider

$$(5.12) \qquad \begin{cases} \Box u + au + f(u) = 0 , & 0 < x < \pi ,\ t \in \mathbb{R} \\ u(0,t) = 0 = u(\pi,t) , & u(x,t+2\pi) = u(x,t) \end{cases} .$$

Theorem 5.13: If $a \geq 0$ and f satisfies (f_1)-(f_3), then (5.12) possesses a nontrivial solution $u = v+w$ with $v \in C^1 \cap \mathbb{N}$, $w \in C^2 \cap N^\perp$.

Proof: There are several approaches one could take to this problem such as beginning anew with a variant of (1.8) in which βv_{tt} is replaced by a term more suitable to (5.12). However we will remain in the framework set up in §1-4 in order to take advantage of the theory we have developed. The first significant modification to our earlier arguments arises in the proof of Lemma 1.13 where we must replace N, $N^\pm$ by other spaces. Let

$$N_1 = \text{span}\,\{\sin jx \sin kt,\ \sin jx \cos kt \mid k^2 = a + j^2\} ,$$

$$N_2^+ = \text{span}\,\{\sin jx \sin kt,\ \sin jx \cos kt \mid k^2 > a + j^2\} , \qquad \text{and}$$

$$N_2^- = \text{span}\,\{\sin jx \sin kt,\ \sin jx \cos kt \mid k^2 < a + j^2\} .$$

Observe that $N_1 = \{0\}$ is possible. Set $\mathbb{R}^{k+1} \equiv E_n \cap N_1 \oplus N_2^- \oplus \text{span}\{\sin x \sin \bar{k} t\}$ where $\bar{k}$ is the smallest element of $\mathbb{N}$ satisfying $\bar{k}^2 > a + 1$. Then the conclusion of Lemma 1.11 obtains with this choice of $\mathbb{R}^{k+1}$. Moreover the argument of Lemma 1.14 goes through essentially as earlier with (2^2-1) replaced by $\bar{k}^2 - (a+1)$. The presence of the a term does not effect any of the estimates and arguments of §2-4 required for the existence of a solution of (5.12) in an unfavorable fashion. (See also Remark 1.15 in this regard). Thus we get existence of a solution

$u(\beta, K)$ of (1.8) (with $f_K \to au + f_K$) and u of (5.12). The proofs of Lemmas 2.47 and 3.40 are not valid here since they employed (f_3)(i) which is not satisfied by $a z + f(z)$. Lemma 2.47 can be salvaged as follows: The analogue of (2.49) for our equation (with $v_n \in N$, $w_n \in N^\perp$ as usual) is

$$\gamma_1 \beta \|v_n\|_C^2 \le \beta \|v_{nt}\|_{L^2}^2 + a\|v_n\|_{L^2}^2 \le \gamma_2 \|f_K(u_n)\|_C \|v_n\|_C . \tag{5.14}$$

This observation gives (2.51), Λ_n being defined as earlier, with different constants:

$$\beta \|v_n\|_C \le \gamma_3 \varepsilon \Lambda_n + \gamma_4 \Lambda_n^3 . \tag{5.15}$$

Now we distinguish between two cases depending on whether N_1 is trivial or not.

<u>Case 1:</u> $\dim N_1 = 0$.

Then (2.15) implies $(\Box + a)^{-1}$ is continuous from $L^2 \cap N^\perp \to C \cap N^\perp$ so

$$\|w_n\|_C \le \gamma_5 \|-\beta v_{ntt} + a v_n + P_n f_K(u_n)\|_{L^2} \tag{5.16}$$

and by (2.6), (5.14), and (2.50),

$$\|w_n\|_C \le \gamma_6 (\varepsilon \Lambda_n + A_\varepsilon \Lambda_n^3) \tag{5.17}$$

Combining (5.15) and (5.17) as in Lemma 2.47 then shows Λ_n is bounded away from 0 .

<u>Case 2:</u> $\dim N_1 > 0$.

We argue indirectly for this case using a comparison argument. To denote the dependence of N_1, $N_2^\pm$ on a, we write $N_1(a)$, etc. Observe that for $\alpha > a$, $N_1(a) \oplus N_2^-(a) \subset N_2^-(\alpha)$. We have $\dim N_1(a) > 0$. For α slightly larger than a, $\dim N_1(\alpha) = 0$ and $N_1(a) \oplus N_2^-(a) = N_2^-(\alpha)$. We fix such an α. Let

$$I_\lambda(u) = \int_T [\tfrac{1}{2}(u_t^2 - u_x^2 - \beta v_t^2 - \lambda u^2) - F_K(u)]dxdt . \tag{5.18}$$

Then $I_a(u) \ge I_\alpha(u)$ for all $u \in E_n$ and from Lemma 1.13 we see $\Gamma_n(a) = \Gamma_n(\alpha)$ and $c_n(a) \ge c_n(\alpha) > 0$. For ε, A_ε as in (2.50), define

$$J_\lambda(u) = \int_T [\tfrac{1}{2}(u_t^2 - u_x^2 - \beta v_t^2 - \lambda u^2 - \varepsilon u^2) - \tfrac{1}{4} A_\varepsilon u^4]dxdt . \tag{5.19}$$

Then as in Lemma 3.40, for ε sufficiently small compared to $\alpha - a$, we have a critical value $b_n(\alpha)$ of $J_\alpha(u)$ with $c_n(\alpha) \ge b_n(\alpha) > 0$.

Suppose $\|u_n(a)\|_C \to 0$ as $n \to \infty$. Then by (1.28), $C_n(a)$ and *a fortiori* $b_n(\alpha) \to 0$. This implies $\|\bar{u}_n(\alpha)\|_{L^4} \to 0$ by (3.45). But the argument of Case 1 (with ε possibly still smaller) shows $\|\bar{u}_n(\alpha)\|_C$ is bounded away from 0. Since $\bar{u}_n(\alpha)$ converges in C, this is contrary to (3.45).

Thus we have shown $u(\beta, K)$ is nontrivial. A more complicated argument based on the above and the proof of Lemma 3.40 then shows the solution u of (5.12) is nontrivial. We will not carry out the details.

For our final results, we consider

$$(5.20) \qquad \begin{cases} \Box u + f(x,t,u) = 0, & 0 < x < \pi,\ t \in \mathbb{R} \\ u(0,t) = 0 = u(\pi,t), & u(x,t+2\pi) = u(x,t) \end{cases}$$

first under the hypotheses:

$(\bar{f}_1)$ $f \in C^1([0,\pi] \times R^2, \mathbb{R})$, $f(x,t,0) \equiv 0$, and f is 2π periodic in t

$(\bar{f}_2)$ $f(x,t,z)$ is strictly monotonically increasing in z and $f_z(x,t,z) > 0$ for z near 0, $z \neq 0$

$(\bar{f}_3)$ (i) $f(x,t,z) = o(|z|)$ at $z = 0$

(ii) There is a $\bar{z} > 0$ so that $F(x,t,z) = \int_0^z f(x,t,s)ds \le \theta z f(x,t,z)$ for $|z| \ge \bar{z}$ where $\theta \in [0, \frac{1}{2})$.

Here we can show

Theorem 5.21: If f satisfies $(\bar{f}_1)$-$(\bar{f}_3)$, (5.20) possesses a nontrivial solution $u = v + w \in (C^1 \cap N) \oplus (C^2 \cap N^{\perp})$.

Proof: A truncation of f analogous to (1.7) is required. Let

$$(5.22) \qquad f_K(x,t,z) = \begin{cases} f(x,t,z) \text{ if } |z| \le K, \\ f(x,t,K) + f_z(x,t,K)(z-K) + \rho(K)(z-K)^3, & z > K \\ f(x,t,-K) + f_z(x,t,-K)(z+K) + \rho(K)(z+K)^3, & z < -K. \end{cases}$$

Note that f_K is merely continuous in (x,t) for $|z| > K$. It is easy to verify that the results of §1-2 are valid for this case with only minor modifications until Lemma 2.32 where it was necessary to differentiate f_K with respect to t. This we cannot do unless $\|u_n(\beta,K)\|_C \le K$. However Remark 2.57 gives us a weak solution of the modified equation for (5.20) with $v \in C^2$ and $w \in C^1$. Since (3.2), (3.6), and (3.8) are still valid, we get $\|u(\beta,K)\|_C \le A_5 + M_5$ independently of β and K. Hence for $K > A_5 + M_5$, and for n appropriately large, since

$u_n(\beta,K) \to u(\beta,K)$ in C, we can assume $\|u_n(\beta,K)\|_C \leq K$. By our above observation, Lemma 2.32 is now applicable for n large. Thus the proof proceeds essentially as earlier with the second part of hypotnesis ($\bar{f}_2$) now required for Case 2 of the proof of Tneorem 4.1.

Remark 5.23: The observations made in Remark 4.15 apply equally well to (5.20).

As our concluding result we have

Theorem 5.24: If $-f$ satisfies ($\bar{f}_1$)-($\bar{f}_3$), (5.20) possesses a nontrivial solution $u = v+w \in (C^1 \cap N) \oplus (C^2 \cap N^\perp)$.

Proof: The proof of Theorem 5.21 works for this case with the following changes: In the definition of I, replace β by $-\beta$. Then use $-I$ in the minimax argument of Lemma 1.13 and also reserve the roles of N^+ and N^-.

Remark 5.25: Theorem 5.24 applies equally well when f is independent of t. However for this case our methods do not guarantee that the solution $u(x,t)$ depends explicitly on t, i.e. we may only obtain a solution of the ordinary differential equation

$$(5.26) \qquad \begin{cases} \dfrac{-d^2u}{dx^2} + f(x,u) = 0, & 0 < x < \pi \\ u(0) = 0 = u(\pi). \end{cases}$$

Appendix

The purpose of this Appendix is to prove Theorem 1.10. To do so, we require two preliminaries: a standard result from the calculus of variations and a topological intersection lemma. We are indebted to E. Fadell for the proof of the latter.

Lemma A.1: Let $J \in C^1(\mathbb{R}^m, \mathbb{R})$, $A_s = \{x \in \mathbb{R}^m \mid J(x) \leq s\}$, and $K_s = \{x \in \mathbb{R}^m \mid J(x) = s$ and $J'(x) = 0\}$. Let $\bar{\varepsilon} > 0$ and $c \in \mathbb{R}$. Then there is an $\varepsilon \in (0, \bar{\varepsilon})$ and an $\eta \in C([0,1] \times \mathbb{R}^m, \mathbb{R}^m)$ such that

1° $\qquad \eta(t,x) = x \quad \text{if} \quad J(x) \notin (c-\varepsilon, c+\varepsilon)$

2° $\qquad \eta(1, A_{c+\varepsilon}) \subset A_{c-\varepsilon} \quad \text{if} \quad K_c = \phi$.

For a proof of this lemma see e.g. [7] or [12]. The notation of Theorem 1.10 will be employed in the next result.

Lemma A.2: Let $b_R = \mathbb{R}^{k+1} \cap \bar{B}_R$ and $b_R^+ = \{x \in \bar{B}_R \mid x_{k+1} \geq 0\}$. If $g \in C(b_R^+, \mathbb{R}^m)$, $m > k$, $\rho < R$, and there is a homotopy $G \in C([0,1] \times \partial b_R^+, \mathbb{R}^m \setminus (\partial B_\rho \cap (\mathbb{R}^k)^\perp))$ such that $G(0,x) = x$ and $G(1,x) = g$, then $g(b_R^+) \cap \partial B_\rho \cap (\mathbb{R}^k)^\perp \neq \phi$.

Proof: Intersection theory in $\mathbb{R}^m$ [13, p. 197] provides a simple proof. To normalize the problem somewhat, let $R = 1$, $\rho < 1$. Set $D^{m-k} = \bar{B}_\rho \cap (\mathbb{R}^k)^\perp$, $\partial D^{m-h} = \partial B_\rho \cap (\mathbb{R}^k)^\perp$, $b^+ \equiv b_1^+$, and $\partial b^+ = \{x \in b^+ \mid x_{k+1} = 0 \text{ or } |x| = 1\}$. The intersection pairing

$$H_{m+k}(D^{m-k}, \partial D^{m-k}) \times H_k(\partial b^+) \to \mathbf{Z}$$

yields +1 as intersection number for appropriately chosen generators in the homology groups above. Furthermore naturality yields the diagram

$$\begin{array}{ccl} H_{m+k}(D^{m-k}, \partial D^{m-k}) \times H_k(\partial b^+) & \to & \mathbf{Z} \\ \Big\downarrow \mathrm{id} \times j_* & \nearrow & \\ H_{m+k}(D^{m-k}, \partial D^{m-k}) \times H_k(\mathbb{R}^m \setminus \partial D^{m-k} & & \end{array}$$

where $j: \partial b^+ \subset \mathbb{R}^m \setminus \partial D^{m-k}$ is inclusion. If $g \in C(b^+, \mathbb{R}^m \setminus \partial D^{m-k})$ and G as in the statement of the Lemma, then j_* would be the trivial homomorphism and thus would force the intersection number to be 0 which is a contradiction.

Theorem 1.10: Let $J \in C^1(\mathbb{R}^m, \mathbb{R})$ and $I: \mathbb{R}^m \to \mathbb{R}$ where $J(x) \le I(x)$ for all $x \in \mathbb{R}^m$ and $J(x) \le 0$ for all $x \in \mathbb{R}^k$. If there are constants $R > r > 0$ such that $J > 0$ in $(B_r \setminus \{0\}) \cap (\mathbb{R}^k)^\perp$ while $I \le 0$ on $\mathbb{R}^m \setminus B_R$, then J has a critical point in $\{x \in \mathbb{R}^m \mid J(x) > 0\}$ and a corresponding critical value characterized by

$$(1.11) \qquad c = \inf_{h \in \Gamma} \max_{x \in \mathbb{R}^{k+1} \cap \bar{B}_R} J(h(x)) > 0$$

where $\Gamma = \{h \in C(\mathbb{R}^{k+1} \cap \bar{B}_R, \mathbb{R}^m) \mid h(x) = x \text{ if } I(x) \le 0\}$.

Proof: Let $h \in \Gamma$. Then $h(x) = x$ if $I(x) \le 0$. Since $I \le 0$ on $\mathbb{R}^k \cup \partial B_R$, $h(y) = y$ on ∂b^+_R. Hence $h = g$ trivially satisfies the homotopy hypothesis in Lemma A.2 for all $\rho < R$. Therefore for all such ρ,

$$(A.3) \qquad h(b^+_R) \cap \partial B_\rho \cap (\mathbb{R}^k)^\perp \ne \emptyset .$$

Choosing any $\rho < r$ and letting

$$\alpha = \min_{z \in \partial B_\rho \cap (\mathbb{R}^k)^\perp} J(z) ,$$

then α is positive by hypothesis and by (A.3) with $\rho = r$

$$(A.4) \qquad \max_{x \in \mathbb{R}^{k+1} \cap \bar{B}_R} J(h(x)) \ge \alpha > 0 .$$

Thus by (A.4) and the definition of c, $c \ge \alpha > 0$. It remains to show that c is a critical value of J. We argue indirectly. Suppose c is not a critical value of J. Then we can invoke Lemma A.1 with $\bar{\varepsilon} < \frac{c}{2}$. Observe that if $h \in \Gamma$, then $\eta(1,h) \in C(\mathbb{R}^{k+1} \cap \bar{B}_R, \mathbb{R}^m)$. Moreover if $I(x) \le 0$, then $J(x) \le 0$ and $h(x) = x$. Therefore by 1° of Lemma A.1 and our choice of $\bar{\varepsilon}$, $\eta(1,h(x)) = \eta(1,x) = x$. Consequently $\eta(1,h) \in \Gamma$.

Finally choose $h \in \Gamma$ such that

$$(A.5) \qquad \max_{x \in \mathbb{R}^{k+1} \cap \bar{B}_R} J(h(x)) \le c + \varepsilon .$$

Then by 2° of Lemma A.1,

$$(A.6) \qquad \max_{x \in \mathbb{R}^{k+1} \cap \bar{B}_R} J(\eta(1,h(x)) \le c - \varepsilon ,$$

contrary to the definition of c. Thus the theorem is proved.

Remark A.7: If $c = \max_{B_R} J$, a refined version of Theorem 1.10 shows K_c contains infinitely many critical points. One can also give an infinite dimensional version of the theorem. Since these results are extraneous to the main topic of this paper, we will pursue them elsewhere.

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Existence and regularity of nontrivial time periodic solutions are proved for semilinear wave equations of the form

$$u_{tt} - u_{xx} + f(x,u) = 0 \qquad 0 < x < \pi,\ t \in \mathbb{R}$$
$$u(x,b) = 0 = u(\pi,b)$$

under mild smoothness, monotonicity, and superlinearity assumptions ~~on f~~. The forced vibration case where f depends on b in a time periodic fashion is also treated.

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